Drawing and Visualisation Research

Published in *TRACEY* | journal
Thinking
December 2014

www.lboro.ac.uk/departments/sota/tracey/
tracey@lboro.ac.uk

Special Edition:
Drawing in STEAM

DRAWING AND MATHEMATICS: GEOMETRY, REASONING, AND FORM

Gemma Anderson [a], Dr Dorothy Buck [b], Dr Tom Coates [b], Prof. Alessio Corti [b]

[a] Falmouth University
[b] Imperial College London

In this paper we consider various ways that drawing occurs in mathematics. We describe, and give examples of, drawing-based mathematical proof: in this context drawing is a language for communicating mathematical reasoning. We then describe our artistic collaboration, where drawing functions both as a language for interdisciplinary communication, essential to the formative process, and as the artwork itself.

The literature on the connections between mathematics and the visual arts tends to emphasize the mathematics and aesthetics of proportion and form, through for example the Golden Ratio $\varphi = \frac{1}{2}(1+\sqrt{5})$ and the Fibonacci Series, or to track the influence of mathematical geometries on artists such as Dorothea Rockburne and Naum Gabo. In this paper we consider a quite different connection between mathematics and the arts, which has so far been overlooked: the role of drawing in mathematical proof. This is one of several ways in which drawing occurs in research mathematics. As we have argued elsewhere (Anderson et al 2014), mathematicians and artists both use drawing as a way of coming to know and understand the world—indeed this shared way of knowing has been crucial to our mathematical/artistic collaboration. These new and underexplored connections between mathematics and visual art merit a careful analysis.

Drawing as Mathematical Proof

A mathematical proof is a step-by-step sequence of deductions, where each step is a logical consequence of the step preceding it. This sequence starts from something that is known to be true and ends with the statement to be proved. We introduce and illustrate the notion of drawing-based mathematical proof by giving an example: a drawing-based proof of the famous Theorem of Pythagoras.

THEOREM OF PYTHAGORAS

> *In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.*

What does this mean? "The hypotenuse" is the longest side of a right-angled triangle; this is always the side opposite the right angle. So the Theorem of Pythagoras is the assertion that if the sides of a right-angled triangle are of lengths *a, b,* and *c* as shown:

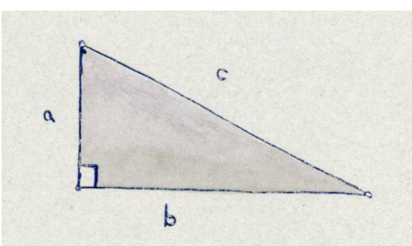

FIG. 1

then $a^2+b^2=c^2$.

Let us prove this. First, consider a square with side-length $a+b$, divided as shown in Figure 2. The left-hand shaded square in Figure 2 has side-length a, and hence area a^2. The right-hand square in Figure 2 has side-length b, and hence area b^2. Each of the four triangles in Figure 2 is a right-angled triangle with the two shorter sides having lengths *a* and *b*. Thus each of the four triangles in Figure 2 is a copy of the triangle shown in Figure 1; in particular, therefore, the hypotenuse (longest side) in each of the four triangles has length *c*.

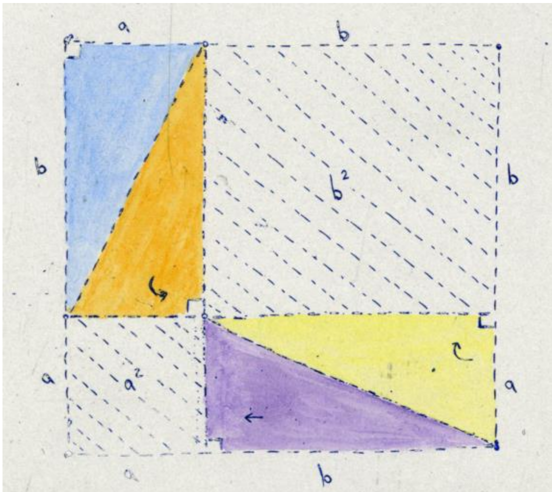

FIG. 2

Now consider a square of side-length *a+b* but divided differently, as shown in Figure 3. Each of the four triangles in Figure 3 is (once again) a right-angled triangle with the two shorter sides having lengths *a* and *b*. Thus each of the four triangles in Figure 3 is (once again) a copy of the triangle shown in Figure 1. The shaded square in Figure 3 therefore has side-length *c*, and area c^2.

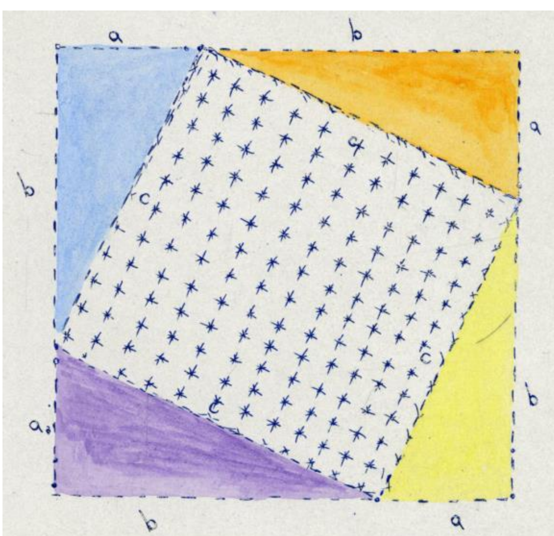

FIG. 3

Now the total shaded area in Figure 2 is equal to the total shaded area in Figure 3, as they are each equal to the area of the big square minus the area of four copies of the triangle from Figure 1. But the total shaded area in Figure 2 is a^2+b^2, and the total shaded area in Figure 3 is c^2. We conclude that $a^2+b^2=c^2$. QED

Drawings, Diagrams, and Diagrams

The drawings that formed the basis of our proof of the Theorem of Pythagoras are what might more typically be called *diagrams*. By diagram here we mean:

> *an illustrative figure which, without representing the exact appearance of an object, gives an outline or general scheme of it, so as to exhibit the shape and relations of its various parts (OED online)*

Our next example of a drawing-based proof will involve a branch of mathematics called knot theory. In knot theory, the word *diagram* has a technical meaning: it means a picture (or, more accurately, a projection) of a mathematical knot. But, as we will explain below, only the crude shape of these pictures is important. One should think of the figures in the next section as drawings of imaginary objects (as in Figure 4) or as hints as to how to manipulate these objects within your mind (as in Figure 6). We will refer to the figures as *diagrams*, following customary usage in knot theory, but the reader should be aware that the meaning of this word has changed.

UNKNOTTING NUMBER

For a mathematician, a knot is a closed curve in three-dimensional space, which can twist around in any way that you like but which never crosses itself. By "closed curve" we mean "curve with no ends". In other words, if you were to make a mathematical knot from a piece of string then you should finish by sealing the two ends of the string together. Here are two examples:

FIG. 4

The diagram on the left is a picture of the simplest possible mathematical knot, called the *unknot*. The diagram on the right is a picture of the second-simplest knot, called the *trefoil*. Notice how the crossings of the diagram show how the curve making up the knot passes over or under itself.

Mathematicians study knots for many reasons. One of us (Dr Dorothy Buck) studies how DNA molecules become knotted and linked during cellular processes such as replication and recombination, and how these changes in form affect the biology of the cell. Thus knot theory is of interest to mathematical biologists. But knot theory started as, and remains, an important part of the field of *topology*: the mathematical study of shape[1].

We regard two knots as the same if you can smoothly deform one of them into the other, without breaking the curve or pushing it through itself. For example, the knot pictured in Figure 5

FIG. 5

is actually the unknot: to see this imagine pulling the loop in the middle tight, and then untwisting it. It is much easier to draw this sequence of transformations than to describe it in words — see Figure 6.

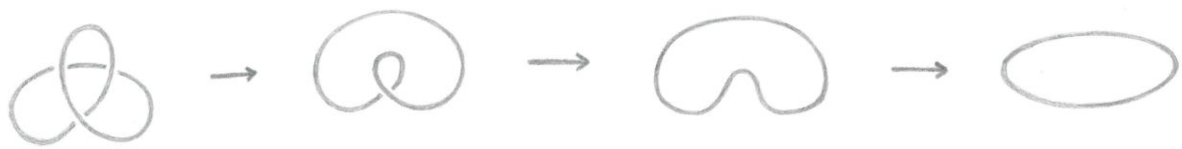

FIG. 6

Figures 4, 5 and 6 also illustrate an important point: just because two pictures of a knot are different does *not* mean that the knots themselves are different. The knot in Figure 5 and the knot pictured in the left-hand diagram in Figure 4 are the same even though the diagrams are different. The fact that these two different diagrams represent the same knot is demonstrated in Figure 6.

We now turn to our second example of a drawing-based mathematical proof. For this we need to introduce a new concept, *unknotting number*. The unknotting number of a knot is a measure of the complexity of that knot. It is the minimum number of times that you need to push the curve through itself in order to turn it into the unknot. In terms of a diagram of the knot, pushing the curve through itself corresponds to turning an undercrossing into an overcrossing, or *vice versa*:

[1] Dr Buck is both a topologist and a mathematical biologist.

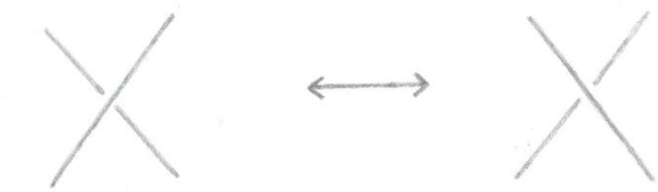

FIG. 7

The following diagram shows that the unknotting number of the trefoil is one: if we take the standard picture of the trefoil and change one of the crossings[2] from an overcrossing into an undercrossing or *vice versa* then you get the unknot.

FIG. 8

Let us finish this section with a more complicated example: we will prove that the unknotting number of the knot called 8_{10}, which is shown in Figure 9, is two.

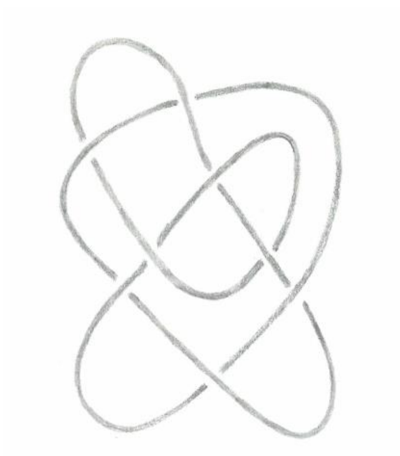

FIG. 9

First, change the crossing indicated from an undercrossing to an overcrossing (or in other words, push the curve through itself once in the place shown).

---

[2] It does not matter which of the three crossings you change. In each case, you get the unknot.

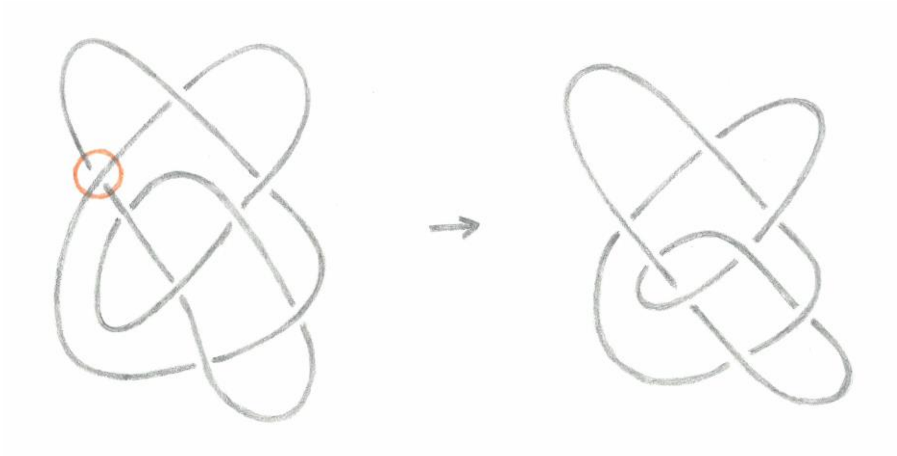

FIG. 10

Next deform the knot as shown in Figure 11.

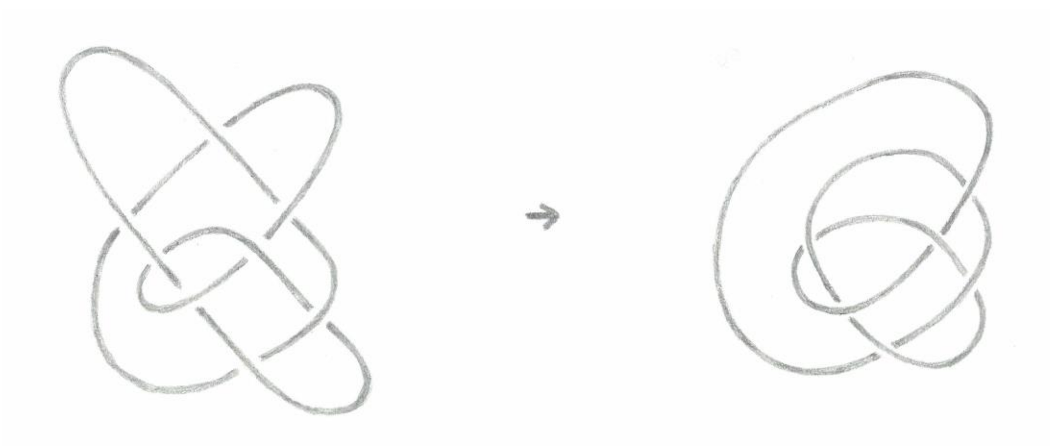

FIG. 11

Now change the crossing indicated from an undercrossing to an overcrossing (or in other words, push the curve through itself in the place indicated; this is the second time we have pushed the curve through itself).

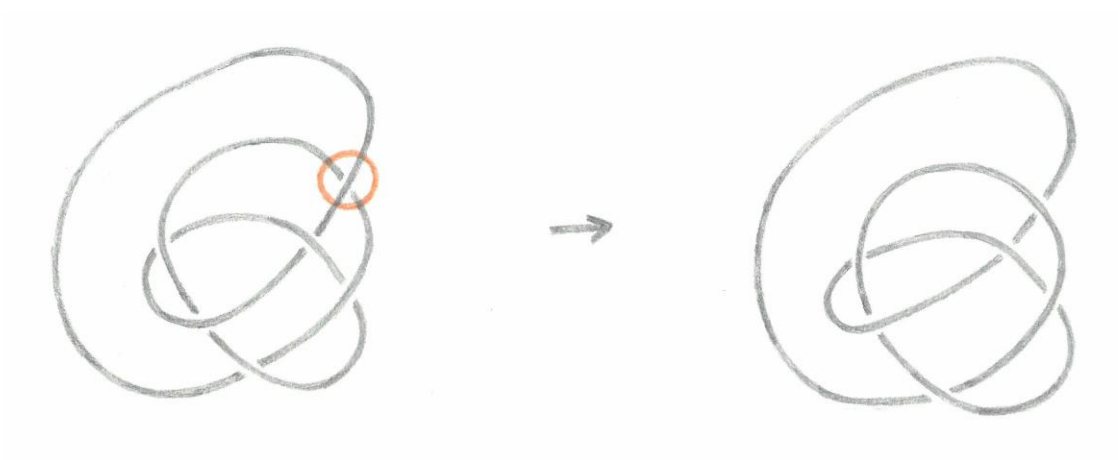

FIG. 12

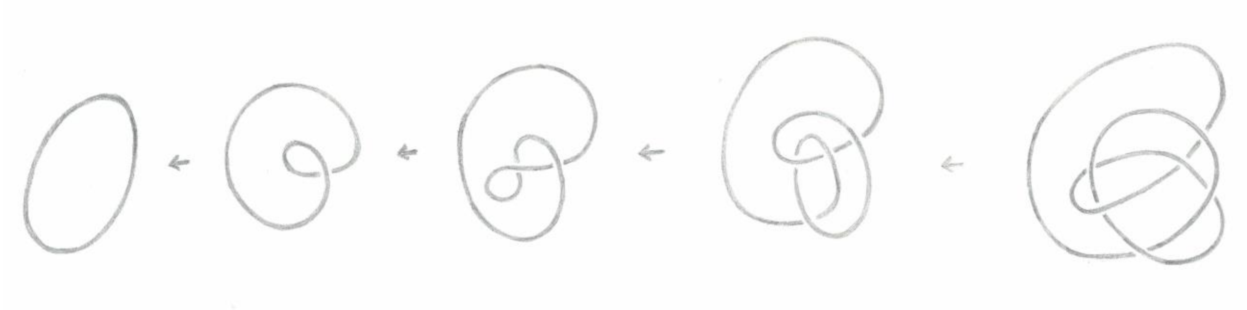

FIG. 13

Finally, deform the knot as shown in Figure 13. The end result is the unknot. Thus we have given a drawing-based proof that we can unknot 8_{10} by making two crossing changes.

Let us close this section by pointing out a subtlety. We have shown that we can unknot 8_{10} by making two crossing changes. This shows that the unknotting number of 8_{10} is at most two. But to show that the unknotting number of 8_{10} is exactly two we need to show that two is the *minimum* number of crossing changes required: in other words, we need to show that there is *no* diagram of a knot such that making a single crossing change will turn 8_{10} into the unknot[3]. This is substantially harder: it requires the full power of the Osváth–Szabó theory of Heegaard Floer knot homology (Osváth and Szabó 2005).

Our Collaboration

We now turn to our artistic collaboration. This began in 2011, when Anderson found herself reading the article ‘A Periodic Table of Shapes’ in the Imperial College Newsletter. The article described the research of Tom Coates and Alessio Corti, who study geometric forms called Fano Varieties that are “atomic pieces” of mathematical shapes.

Anderson immediately took the article back to her studio and began making drawings, exploring the Fano forms. This subsequently developed into a full collaboration, first with Coates and Corti and then later also with Dorothy Buck.

Drawing has played an essential role in our project. During hundreds of conversations about scientific ideas — about string theory, hyperbolic geometries, polyhedra, topology,

[3] This issue did not arise when we showed that the unknotting number of the trefoil knot is one. The trefoil and the unknot are different, so the unknotting number of the trefoil is at least one. And we showed that there is a diagram where making one crossing change turns the trefoil into the unknot. So the unknotting number of the trefoil is exactly one.

knot theory, DNA, and many other topics --- drawings have formed the bridge that allows interdisciplinary communication.

These drawings are largely informal, notational, and schematic (see Figures 14 and 15). They accompany and form an integral part of conversations, with drawing functioning as a non-verbal, intuitive language for scientific concepts. The precise role of drawing differs from place to place in the conversation: communicating the visualization needed for understanding; sharpening these visualizations; or creating understanding (for the drawer) through the physical act of drawing.

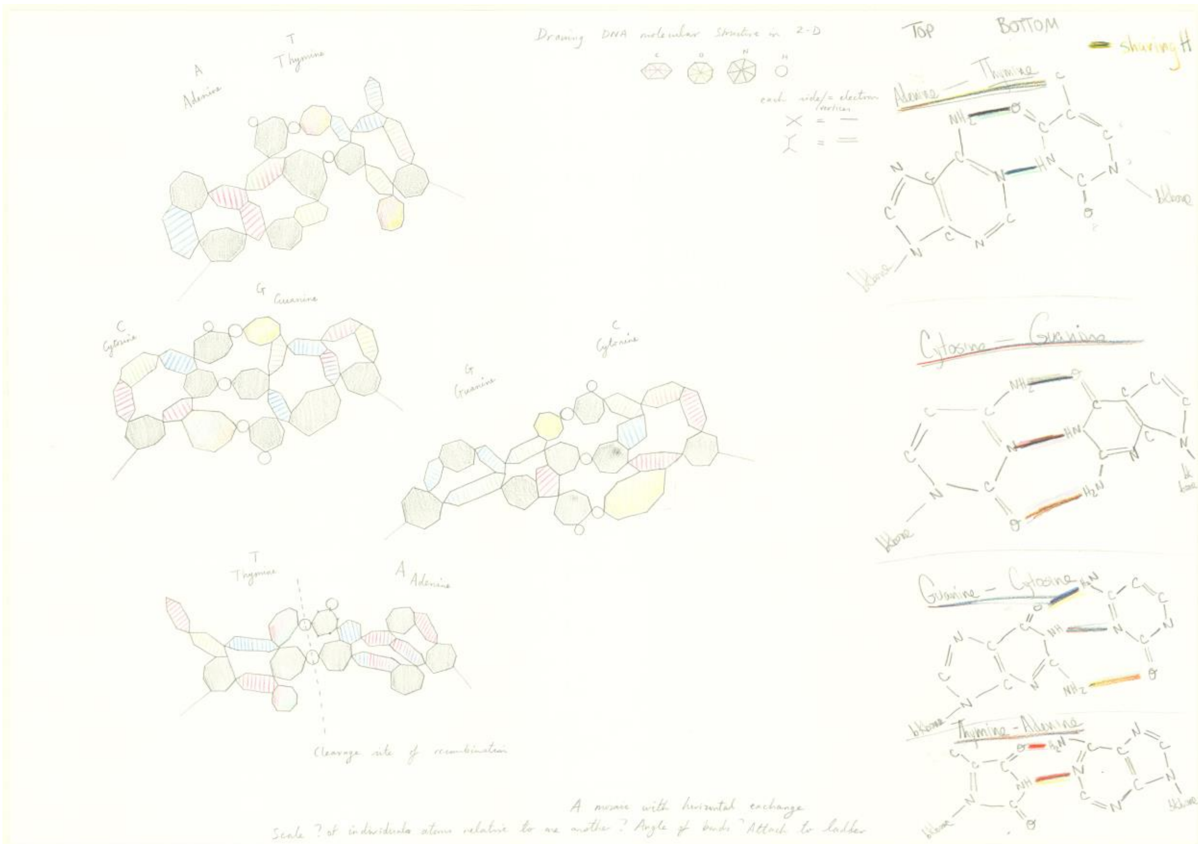

FIG. 14

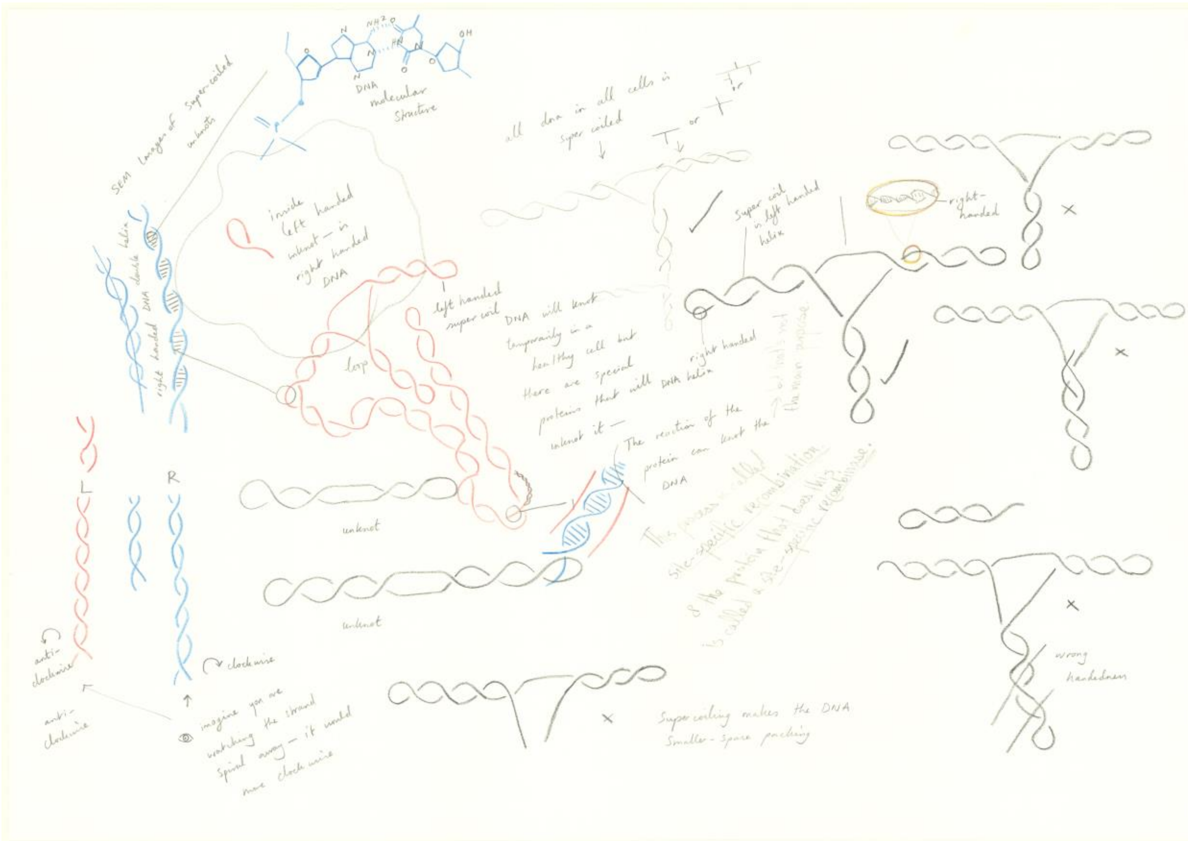

FIG. 15

Drawing also plays a different, and deeper, role in our collaboration. Because the creative processes of the mathematicians involved are heavily visual and drawing-based, Anderson can witness and connect to the process of doing mathematical research; this directly inspires artworks based on the geometries and forms involved (Figure 16).   Anderson in turn responds with unique insights and resonances, the result of her practiced observational drawing across the natural world. The works that we create thus admit multiple overlapping perspectives, holding within them as they do the different logics of the artist and the mathematician.

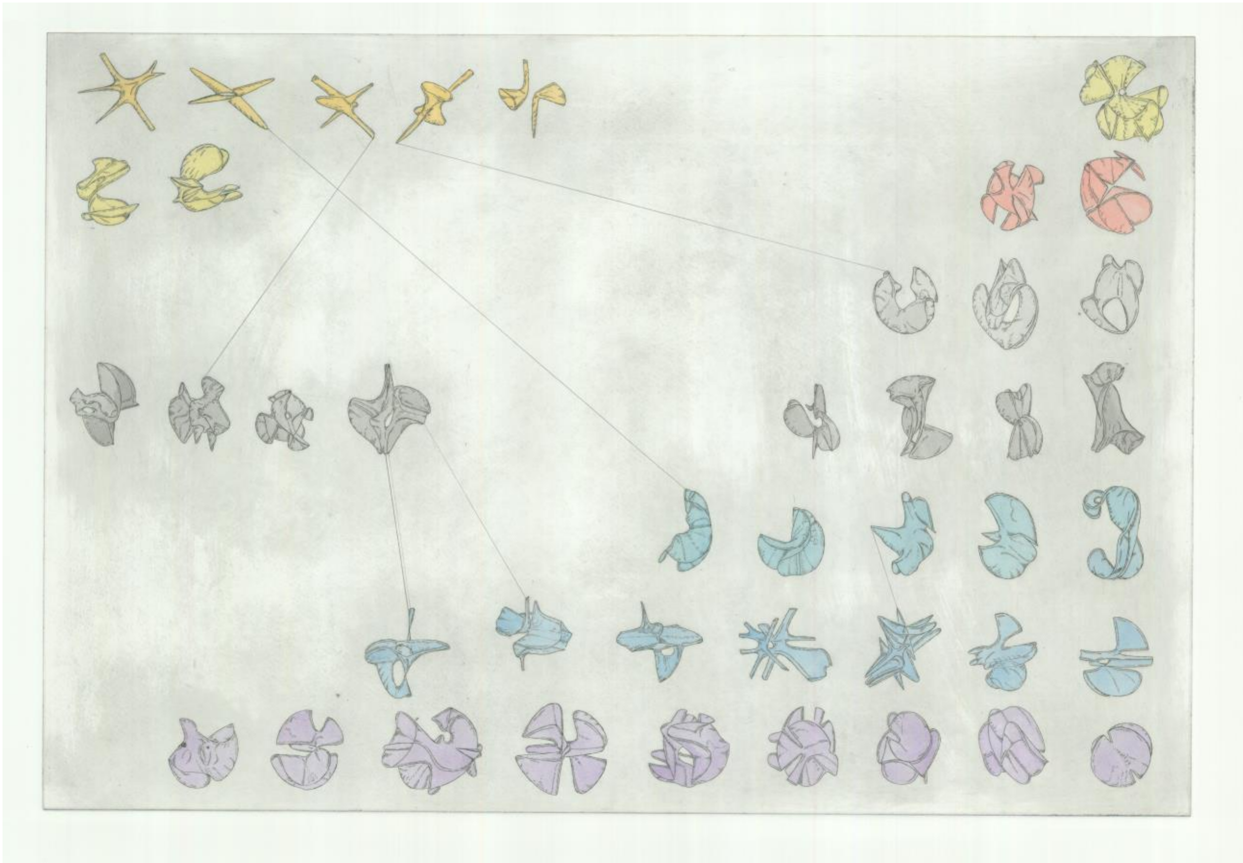

FIG. 16

REFERENCES

ANDERSON, Gemma, Dorothy BUCK, Tom COATES and Alessio CORTI. 2014. 'Drawing in Mathematics: from Inverse Vision to the Liberation of Form' to be published in *Leonardo.*

OED Online. 2014. 'Diagram'. Oxford University Press[online]. Accessed 24 July 2014.

Ozsváth, P. and Z Szabó. 2005. 'Knots with unknotting number one and Heegaard Floer homology'. *Topology* 44 (2005), no. 4, 705–745.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge financial support from the Engineering and Physical Sciences Research Council, the Royal Society, and the Leverhulme Trust. We thank Dr Alexander Kasprzyk, Andrew MacPherson, Simon Bird, and Adam Springer for valuable technical assistance.